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Partition functions for membrane theories

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Abstract: Partition functions for M2-brane theories in various backgrounds are computed. We consider in particular configurations of membranes at orbifold singularities preserving $\mathcal{N} = 5$ or $\mathcal{N} = 6$ supersymmetry. The worldvolume membrane theory for some of these configurations has been recently constructed in terms of $\mathcal{N} = 6$ Chern-Simons theories. The detailed structure of the partition functions as well as their transformation rules under the R-symmetry are explicitly computed using the Plethystic Programme.

Keywords: Supersymmetric gauge theory, Brane Dynamics in Gauge Theories, M-Theory.

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Contents

1. Introduction

Partition functions for BPS operators in supersymmetric field theories are rather interesting objects in many respects. They share information about the structure of the moduli space of vacua and the effective number of degrees of freedom in the system. The computation of such generating functions is generically a very hard problem but it can be simplified in particular circumstances. The partition functions for chiral operators in four dimensional supersymmetric gauge theories have been extensively studied in the past years, ranging from SQCD [1–5] to quiver gauge theories living on branes at singularities [6–15]. The latter in particular are superconformal gauge theories and have an $AdS_5 \times H$ dual [16–18]. In this case, information from the field theory and from the holographic dual can be combined to give a better understanding of the superconformal theory. For the case of branes at Calabi-Yau singularities, the combination of the Plethystic Programme with algebraic tools in complex geometry allows to write quite explicit formulae for the partition functions.

It is a natural and interesting direction to try and extend these results to other dimensions. In particular, the case of three dimensions, where the superconformal zoo is very large, is a natural choice. Most supersymmetric Yang-Mills theories flow in the IR to a superconformal fixed point in three dimensions. However, for theories with an $AdS_4 \times H$

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dual, it is very difficult to write the corresponding UV Yang-Mills theory[1] and little is known about the explicit description of the interacting superconformal theory which is assumed to be a theory of membranes.

In this paper we will consider theories of membranes living at singularities $\mathbb{R}^8/\Gamma$ and preserving $\mathcal{N}=5$ or $\mathcal{N}=6$ supersymmetry. The dual theory is $AdS_4 \times S^7/\Gamma$. Here Γ is any of the discrete subgroups of SU(2) and it acts freely on S^7.

One of the motivation for this analysis is the fact that a superconformal Chern-Simons theory with $\mathcal{N}=6$ supersymmetry and moduli space $\mathbb{R}^8/\mathbb{Z}_k$ has been recently constructed. In fact there was recently much activity in the study of superconformal Chern-Simons theories in three dimensions with large amount of supersymmetry in the attempt of constructing theories for M2-branes. A consistent theory with $\mathcal{N}=8$ supersymmetry has been constructed with gauge group $\mathrm{SU}(2)\times\mathrm{SU}(2)$ in [23] and interpreted as the theory of two M2-branes on an orbifold of flat space for some value of the Chern-Simons parameter [24, 25]. Attempts to extend this construction to N branes and $\mathrm{SU}(N)$ gauge groups keeping manifest $\mathcal{N}=8$ supersymmetry faced intrinsic difficulties in the theory of three Lie algebras. The only available candidates at the moment contain ghosts [26 − 28].[2] However, more recently, a consistent theory with $\mathrm{U}(N)\times\mathrm{U}(N)$ gauge group and bifundamental fields which has only a manifest $\mathcal{N}=6$ supersymmetry has been constructed in [32]. The theory has two parameters, the Chern-Simons parameter k and the number of colors N. Based on the the analysis of the moduli space, the spectrum of chiral operators and a brane construction, this theory has been proposed as the superconformal theory living on N M2-branes at the orbifold singularity $\mathbb{R}^8/\mathbb{Z}_k$. Further evidence of this fact was given in [33, 34]. The theory has a dual description as string theory on $AdS_4 \times S^7/\mathbb{Z}_k$. In particular, for Chern-Simons level $k=1$ we recover the maximally supersymmetric theory of M2-branes in flat space; only an $\mathcal{N}=6$ supersymmetry is however manifest in the Lagrangian.

It is then a natural question to write partition functions for these theories. Of course, many things are known about the chiral spectrum of M2-branes in flat space. A partition function for 1/8 BPS operators was written in [35], for example. For theories with such amount of supersymmetry, it is natural and convenient to write down partition functions which respect the R-symmetry of the superconformal theory, which is Spin(8) for $\mathcal{N}=8$. This can be very efficiently done in the case of one brane.

In fact, it is known from the $AdS_4 \times S^7$ dual description that there is precisely one single trace chiral multiplet for each symmetric traceless representation of Spin(8). Making use of this information, we show how to write the partition function of one M2-brane in flat space and expand it in terms of Spin(8) representations.[3] The supersymmetric partition function on $\mathbb{R}^8/\mathbb{Z}_k$ is then obtained by using the discrete Molien formula (4.1)

[1] We note in passing that the analogous problem in four dimensions is solved, at least for toric Calabi-Yaus, by the dimer technology [19 − 21]. In three dimensions there is a proposal based on crystals [22].

[2] Ghosts can be consistently eliminated but it seems that the resulting theory is $\mathcal{N}=8$ SYM and not its infrared limit [29 − 31].

[3] We remark that, in obtaining the partition function, the Bagger-Lambert or a related theory is not directly applied. Rather, we make use of the expected structure of the moduli space of M2-branes, i.e. $AdS_4 \times S^7/\Gamma$.

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and expanded in terms of representations of the R-symmetry group Spin(6). We will also write the partition function for the theory of an M2-brane on $\mathbb{R}^8/\hat{D}_{k+2}$ and $\mathbb{R}^8/\hat{E}_n$ (with $n = 6,\ 7,\ 8$), configurations that preserves $\mathcal{N} = 5$ supersymmetry and has Sp(2) R-symmetry. We emphasise that the partition functions precisely count chiral multiplets whose lowest compontent is a scalar field.

We can apply the Plethystic Programme in order to get information on the moduli space for higher N, which is $\mathrm{Sym}(\mathbb{R}^8/\Gamma)^N$, where Γ can be the abelian group $\mathbb{Z}_k$ or any of the non-abelian discrete subgroups of SU(2) associated with the affine Dynkin diagrams of D_{k+2}, E_6, E_7, E_8 [8, 40]. Below we write a quick reminder.

The plethystic programme: a recapitulation. Let us define the *plethystic exponential* of a multi-variable function $g(t_1, \ldots, t_n)$ that vanishes at the origin, $g(0, \ldots, 0) = 0$, to be

$$\mathrm{PE}[g(t_1, \ldots, t_n)] := \exp\left(\sum_{r=1}^{\infty} \frac{g(t_1^r, \ldots, t_n^r)}{r}\right) . \tag{1.1}$$

In the same way as mentioned in [8, 11], the generating function g_N at finite N is found by the series expansion of the ν - inserted plethystic exponential as

$$\mathrm{PE}[\nu g_1(t_1, \ldots, t_n)] = \exp\left(\sum_{r=1}^{\infty} \frac{\nu^r g_1(t_1^r, \ldots, t_n^r)}{r}\right) = \sum_{N=0}^{\infty} g_N(t_1, \ldots, t_n)\nu^N . \tag{1.2}$$

Information about the generators of the moduli space and the relations they satisfy can be computed by using the *plethystic logarithm*, which is the inverse function of the plethystic exponential. Using the Möbius function $\mu(r)$ we define

$$\mathrm{PL}[g(t_1, \ldots, t_n)] := \sum_{r=1}^{\infty} \frac{\mu(r) \log g(t_1^r, \ldots, t_n^r)}{r} . \tag{1.3}$$

The significance of the series expansion of the plethystic logarithm is stated in [8, 11]: *the first terms with plus sign give the basic generators while the first terms with the minus sign give the constraints between these basic generators.* If the formula (1.3) is an infinite series of terms with plus and minus signs, then the moduli space is not a complete intersection and the constraints in the chiral ring are not trivially generated by relations between the basic generators, but receive stepwise corrections at higher degree. These are the so-called higher syzygies.

These partition functions can be decomposed into representations of the relevant R-symmetry group, Spin(8), Spin(6) and Spin(5) for $\mathcal{N} = 8, 6$ and 5 supersymmetry respectively. A word of caution is necessary. These partition functions count some gauge invariant multitrace operators. However, since the product of short multiplets of $\mathcal{N} = 8, 6$ and 5 supersymmetry may contain operators that are not protected, the partition functions are not necessarily counting short operators except for $N = 1$. They should be better intended as partition functions counting real functions on the moduli space for N branes. It would be interesting to investigate further the properties of these partition functions and to seek for a dual interpretation for them.

Notation for representations. In this paper, we shall represent an irreducible representation of a group G by its highest weight $[a_1, \ldots, a_r]$, where $r = \text{rank } G$. In order to avoid cluttered notation, we shall also slightly abuse terminology by referring to each character by its corresponding representation.

We next proceed with a detailed study of this class of theories.

2. The theory of $N = 1$ and $k = 1$

This theory has $\mathcal{N} = 8$ supersymmetry in 2+1 dimensions and therefore all protected operators appear in irreducible representations of the R-symmetry group, Spin(8). There is an additional quantum number which counts the number of scalar fields. This quantum number can be taken to be the conformal dimension of the corresponding operators, measured in units of 1/2. Its corresponding fugacity is denoted by t. The moduli space is $\mathbb{R}^8$ and the scalars transform in the $[1,0,0,0]$ representation of Spin(8). It should be noted that the fugacity t represents a real degree of freedom and not a complex degree of freedom. As a result, the dimension of the moduli space is real and not complex. We therefore write down the first partition function for the set of theories $g_{N,\mathbb{Z}_k}$ (below $\mathbb{Z}_1$ means the trivial action),

$$g_{1,\mathbb{Z}_1}(t, y_1, y_2, y_3, y_4; \mathbb{R}^8) = \mathrm{PE}[[1,0,0,0]t]. \tag{2.1}$$

Here and below the notation $[1,0,0,0]$ is taken to be the character of the representation with these highest weights. To be concrete we can choose four complex fugacities y_1, y_2, y_3, y_4 such that

$$[1,0,0,0] = \frac{y_1}{y_2} + y_1 + \frac{y_3 y_4}{y_2} + \frac{y_4}{y_3} + \frac{y_3}{y_4} + \frac{y_2}{y_3 y_4} + \frac{y_2}{y_1} + \frac{1}{y_1}. \tag{2.2}$$

Using this, an explicit expression for $g_{1,\mathbb{Z}_1}$ after evaluating the PE, as defined in (1.1), takes the form

$$\begin{aligned} g_{1,\mathbb{Z}_1}(t, y_1, y_2, y_3, y_4; \mathbb{R}^8) = & \frac{1}{\left(1 - \frac{t}{y_1}\right)(1 - t y_1)\left(1 - \frac{t y_1}{y_2}\right)\left(1 - \frac{t y_2}{y_1}\right)\left(1 - \frac{t y_2}{y_3 y_4}\right)\left(1 - \frac{t y_3}{y_4}\right)} \\ & \times \frac{1}{\left(1 - \frac{t y_4}{y_3}\right)\left(1 - \frac{t y_3 y_4}{y_2}\right)}. \end{aligned} \tag{2.3}$$

The partition function $g_{1,\mathbb{Z}_1}$ has an expansion in terms of characters of Spin(8) as

$$g_{1,\mathbb{Z}_1}(t, y_1, y_2, y_3, y_4; \mathbb{R}^8) = 1 + [1,0,0,0]t + ([2,0,0,0] + [0,0,0,0])t^2 + \cdots \tag{2.4}$$

When we set all the chemical potentials of the Spin(8) symmetry to zero this function takes the form

$$g_{1,\mathbb{Z}_1}(t, 1, 1, 1, 1; \mathbb{R}^8) = \frac{1}{(1-t)^8} = 1 + 8t + 36t^2 + \cdots . \tag{2.5}$$

We first note that this function has a pole of order 8 at $t = 1$ which indicates that the real dimension of the moduli space is 8.

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Operators on S^7. This partition function turns out to count operators which are not protected by supersymmetry, the simplest one being $\mathrm{Tr}(\phi_i\phi_i)$, which is represented by the singlet term in the expansion at order t^2. To cure this we recall that the protected operators are actually in one to one correspondence with harmonic functions on S^7 (see, e.g., [37] for Kaluza-Klein modes on S^7),[4] and the partition function should reflect this condition. It is easily given by a relation which is quadratic in the basic fields and a singlet of Spin(8). We therefore write a partition function for all harmonic functions on S^7,

$$g_{1,\mathbb{Z}_1}(t, y_1, y_2, y_3, y_4; S^7) = (1 - t^2)\mathrm{PE}\left[[1, 0, 0, 0]t\right] . \tag{2.6}$$

This partition function has a nice expansion in terms of characters:

$$g_{1,\mathbb{Z}_1}(t, y_1, y_2, y_3, y_4; S^7) = \sum_{n=0}^{\infty}[n, 0, 0, 0]t^n, \tag{2.7}$$

which indeed reflects the well known fact that harmonic functions at level n on S^7 transform as precisely one copy of the $[n, 0, 0, 0]$ representation of Spin(8). Correspondingly, the $\mathcal{N} = 8$ theory for one M2-brane has a set of protected operators at level n which transform under precisely one copy of the representation $[n, 0, 0, 0]$ of Spin(8). We can further set all Spin(8) chemical potentials to zero and get the expressions

$$g_{1,\mathbb{Z}_1}(t, 1, 1, 1, 1; S^7) = \frac{1 - t^2}{(1 - t)^8} = \frac{1 + t}{(1 - t)^7} = \sum_{n=0}^{\infty}\frac{(n + 3)}{3}\binom{n + 5}{5}t^n, \tag{2.8}$$

The first form indicates that there are 8 generators for S^7 which are subject to 1 relation of order 2. This relation sets the radius of the S^7 to a constant value. The second form indicates that the real dimension of the moduli space is 7. The last form gives the dimensions of the irreducible representations $[n, 0, 0, 0]$ of Spin(8). For reference we quote here the general dimension formula [39] for a generic Spin(8) representation of highest weight $[n_1, n_2, n_3, n_4]$:

$$\dim[n_1, n_2, n_3, n_4] = \frac{\{1\}\{2\}\{3\}\{4\}\{12\}\{23\}\{24\}\{123\}\{234\}\{124\}\{1234\}\{12234\}}{4320}, \tag{2.9}$$

with $\{i\} = n_i + 1$, $\{ij\} = n_i + n_j + 2$, $\{ijk\} = n_i + n_j + n_k + 3$, *etc.*

Decomposing Spin(8) into SU(4) $\times$ U(1). For applications with higher CS level k, we will now rewrite the generating functions in terms of irreducible representations of SU(4), the R-symmetry for $\mathcal{N} = 6$ supersymmetry in 2+1 dimensions. For this purpose we introduce the fugacity b for the baryonic number, and decompose the 8 dimensional representation of SO(8) into two irreducible representations of SU(4):

$$[1, 0, 0, 0]t = [1, 0, 0]t_1 + [0, 0, 1]t_2 \tag{2.10}$$

with the usual relation as borrowed from the conifold partition functions, $t_1 = tb, t_2 = t/b$ (see e.g., [12]). Here t_1 is taken to count the degree of holomorphic functions on $\mathbb{C}^4$ and

[4]References [37, 38] contain an 11 dimensional supergravity argument from which the zero modes for a single M2-brane can be read off.

t_2 counts the degree of anti-holomorphic functions on $\mathbb{C}^4$. Explicit expressions for the characters of the SU(4) representations can be taken to be with 3 complex fugacities, z_1, z_2, z_3 in the form,

$$[1,0,0] = z_1 + \frac{z_2}{z_1} + \frac{z_3}{z_2} + \frac{1}{z_3}, \qquad [0,0,1] = \frac{1}{z_1} + \frac{z_1}{z_2} + \frac{z_2}{z_3} + z_3 \tag{2.11}$$

The generating function $g_{1,\mathbb{Z}_1}$ takes the form

$$g_{1,\mathbb{Z}_1}(t_1, t_2, z_1, z_2, z_3; S^7) = (1 - t_1 t_2)\mathrm{PE}[[1,0,0]t_1 + [0,0,1]t_2]. \tag{2.12}$$

This function has a nice expansion in terms of irreducible representations of SU(4),

$$g_{1,\mathbb{Z}_1}(t_1, t_2, z_1, z_2, z_3; S^7) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[n,0,m]t_1^n t_2^m = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[n,0,m]b^{n-m}t^{n+m}. \tag{2.13}$$

Comparing (2.7) with (2.13), we find the following decomposition:

$$[n,0,0,0]_{\mathrm{Spin}(8)} \to \sum_{r=0}^{n} b^{n-2r}[n-r,0,r]_{\mathrm{SU}(4)} \ . \tag{2.14}$$

A non-trivial check. The dimension formula [39] for a generic representation of SU(4) of highest weights $[n_1, n_2, n_3]$ is

$$\begin{aligned} \dim[n_1,n_2,n_3] &= \frac{\{1\}\{2\}\{3\}\{12\}\{23\}\{123\}}{12} \\ &= \frac{(n_1+1)(n_2+1)(n_3+1)(n_1+n_2+2)(n_2+n_3+2)(n_1+n_2+n_3+3)}{12} \ . \end{aligned} \tag{2.15}$$

This can be used in checking the various relations quoted above and below.

3. $\mathbb{Z}_k$ orbifold actions on the $N = 1$ theory

3.1 The case of $k = 2$

We next turn to the $k = 2$ theories. The R-symmetry is still Spin(8) and we can still count operators using representations of Spin(8). The new ingredient is an orbifold projection on the variable t. Under this orbifold action $t \to -t$ and we need to sum over both sectors, with t and with $-t$. The resulting generating function gets a simple form, restricting to even powers of t,

$$g_{1,\mathbb{Z}_2}(t, y_1, y_2, y_3, y_4; S^7) = \sum_{n=0}^{\infty}[2n,0,0,0]t^{2n} \ . \tag{3.1}$$

Setting the Spin(8) chemical potentials to zero, we find

$$g_{1,\mathbb{Z}_2}(t, 1, 1, 1, 1; S^7) = \frac{1 + 28t^2 + 70t^4 + 28t^6 + t^8}{(1-t^2)^7} \ , \tag{3.2}$$

suitable for a moduli space of real dimension 7.

The plethystic logarithm of the generating function $g_{1,\mathbb{Z}_2}$ is

$$\begin{aligned} \mathrm{PL}\left[g_{1,\mathbb{Z}_2}(t,y_1,y_2,y_3,y_4;S^7)\right] &= [2,0,0,0]t^2 - ([2,0,0,0]+[0,2,0,0]+[0,0,0,0])\,t^4 + \cdots\,, \\ \mathrm{PL}\left[g_{1,\mathbb{Z}_2}(t,1,1,1,1;S^7)\right] &= 35t^2 - 336t^4 + 5376t^6 - 101856t^8 \ldots\ . \end{aligned} \tag{3.3}$$

This indicates that there are 35 basic generators transforming in the SO(8) representation $[2,0,0,0]$ at order t^2, and there are 336 basic relations transforming in the representations $[2,0,0,0]+[0,2,0,0]+[0,0,0,0]$ at order t^4. We note that in this case the moduli space is not a complete intersection, since the plethystic logarithm is an infinite series.

3.2 The case of higher k

For higher values of k the orbifold action does not commute with the Spin(8) R-symmetry group and breaks it to SU(4) with an action on the baryonic charge. The $\mathbb{Z}_k$ orbifold acts on the fugacity b by $b \to wb$, with $w^k = 1$ and we need to sum over all contributions. The result is the following discrete Molien formula (c.f. Equation (3.1) of [8]):

$$g_{1,\mathbb{Z}_k}(t,b,z_1,z_2,z_3) = \frac{1}{k}\sum_{j=0}^{k-1} g_{1,\mathbb{Z}_1}(t,w^j b,z_1,z_2,z_3) \tag{3.4}$$

It is now useful to recall (2.13) and realize that only terms with $n-m=0 \mod k$ survive the projection. We can therefore write an expression for $g_{1,\mathbb{Z}_k}$ as follows:

$$\begin{aligned} g_{1,\mathbb{Z}_k}(t,b,z_1,z_2,z_3;S^7) &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{r=0}^{k-1}[kn_1+r,0,kn_2+r]t_1^{kn_1+r}t_2^{kn_2+r} \\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{r=0}^{k-1}[kn_1+r,0,kn_2+r]b^{k(n_1-n_2)}t^{k(n_1+n_2)+2r}\ . \end{aligned} \tag{3.5}$$

We shall see in examples below that, for an arbitrary CS level k, the generators are in the representations $[1,0,1],[k,0,0]$ and $[0,0,k]$. This is consistent with the analysis of chiral operators performed in [32] for the $\mathcal{N}=6$ CS theory.

An example of $k=2$. As a check, we can recover the previous results for $k=2$. Formula (3.6) gives

$$g_{1,\mathbb{Z}_2}(t,b,z_1,z_2,z_3;S^7) = 1 + \left(b^2[2,0,0]+[1,0,1]+\frac{1}{b^2}[0,0,2]\right)t^2 + \cdots\ . \tag{3.6}$$

Setting $b=z_1=z_2=z_3=1$, we have the unrefined partition function

$$\begin{aligned} g_{1,\mathbb{Z}_2}(t,1,1,1,1;S^7) &= \frac{1+28t^2+70t^4+28t^6+t^8}{(1-t^2)^7} \\ &= 1+35t^2+294t^4+1386t^6+4719t^8+13013t^{10}+\cdots\ . \end{aligned} \tag{3.7}$$

The plethystic logarithm of this expression is

$$\mathrm{PL}\left[g_{1,\mathbb{Z}_2}(t,1,1,1,1;S^7)\right] = 35t^2 - 336t^4 + 5376t^6 - 101856t^8 + \cdots\ . \tag{3.8}$$

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Observe that the coefficient 35 of t^2 in the plethystic logarithm is simply the dimension of the SU(4) representations $[1,0,1]+[2,0,0]+[0,0,2]$ in the second term of (3.7). This indicates that the generators transform in the representations $[1,0,1]$, $[2,0,0]$ and $[0,0,2]$, which is indeed the decomposition of the $[2,0,0,0]$ representation of Spin(8).

An example of $k = 3$. The unrefined partition function is

$$g_{1,\mathbb{Z}_3}(t,1,1,1,1;S^7) = \frac{1-3t+18t^2-10t^3+21t^4+21t^5-10t^6+18t^7-3t^8+t^9}{(1-t)^7(1+t+t^2)^4}$$
$$= 1+15t^2+40t^3+84t^4+240t^5+468t^6+840t^7+\cdots\ . \qquad (3.9)$$

The plethystic logarithm of this expression is

$$\mathrm{PL}\left[g_{1,\mathbb{Z}_3}(t,1,1,1,1;S^7)\right] = 15t^2+40t^3-36t^4-360t^5-492t^6+2880t^7+\cdots\ . \quad (3.10)$$

We note that the coefficient 15 of t^2 is the dimension of the representation $[1,0,1]$, and the coefficient 40 of t^3 is the dimension of $[3,0,0]+[0,0,3]$. This indicates that the generators transform representations $[1,0,1]$, $[3,0,0]$ and $[0,0,3]$.

An example of $k = 4$. The unrefined partition function is

$$g_{1,\mathbb{Z}_4}(t,1,1,1,1;S^7) = \frac{1+12t^2+108t^4+212t^6+358t^8+212t^{10}+108t^{12}+12t^{14}+t^{16}}{\left(1-t^2\right)^7\left(1+t^2\right)^4}$$
$$= 1+15t^2+154t^4+678t^6++2387t^8+6461t^{10}\dots\ . \qquad (3.11)$$

The plethystic logarithm of this expression is

$$\mathrm{PL}\left[g_{1,\mathbb{Z}_4}(t,1,1,1,1;S^7)\right] = 15t^2+34t^4-512t^6+2332t^8+\cdots\ . \qquad (3.12)$$

The coefficient 15 of t^2 is the dimension of the representation $[1,0,1]$. The coefficient 34 of t^4 is the dimension of $[4,0,0]+[0,0,4]-([0,2,0]+[1,0,1]+[0,0,0])$. We note that the correction $[0,2,0]+[1,0,1]+[0,0,0]$, which is contained in the decomposition of $\mathrm{Sym}^2[1,0,1]$, is simply the relation at order t^4. Therefore, the generators transform under the representations $[1,0,1]$, $[4,0,0]$ and $[0,0,4]$.

An example of $k = 5$. The power series of the unrefined partition function is

$$g_{1,\mathbb{Z}_5}(t,1,1,1,1;S^7) = 1+15t^2+84t^4+112t^5+300t^6+560t^7+825t^8+\cdots\ . \quad (3.13)$$

The plethystic logarithm of this expression is

$$\mathrm{PL}\left[g_{1,\mathbb{Z}_5}(t,1,1,1,1;S^7)\right] = 15t^2-36t^4+112t^5+160t^6+\cdots\ . \qquad (3.14)$$

The coefficient 15 of t^2 is the dimension of the representation $[1,0,1]$. The coefficient -36 of t^4 indicates that there are relations transforming in the representation $[0,2,0]+[1,0,1]+[0,0,0]$ at order t^4, as before. The coefficient 112 of t^5 is the dimension of $[5,0,0]+[0,0,5]$. Therefore, the generators transform representations $[1,0,1]$, $[5,0,0]$ and $[0,0,5]$.

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A general formula. The general unrefined partition function which can be obtained from (3.4) is

$$g_{1,\mathbb{Z}_k}(t,1,1,1,1;S^7) = \frac{1}{3(1-t^2)^6(1-t^k)^4} \times \Big[3+27t^2+27t^4+3t^6+\big(-6+11k+6k^2+k^3 -54t^2+27kt^2-6k^2t^2-3k^3t^2-54t^4-27kt^4-6k^2t^4+3k^3t^4-6t^6 -11kt^6+6k^2t^6-k^3t^6\big)t^k+\big(-22k+4k^3-54kt^2-12k^3t^2+54kt^4 +12k^3t^4+22kt^6-4k^3t^6\big)t^{2k}+\big(6+11k-6k^2+k^3+54t^2+27kt^2 +6k^2t^2-3k^3t^2+54t^4-27kt^4+6k^2t^4+3k^3t^4+6t^6-11kt^6-6k^2t^6 -k^3t^6\big)t^{3k}+\big(-3-27t^2-27t^4-3t^6\big)t^{4k}\Big]\,. \tag{3.15}$$

3.3 The $k \to \infty$ limit: restriction to the zero baryonic subspace

In the limit where k goes to infinity, all states with non zero baryonic charge disappear from the spectrum. We obtain a partition function which counts real functions on $\mathbb{P}^3$,

$$g_{1,\mathbb{Z}_k}(t,z_1,z_2,z_3;\mathbb{P}^3) = \sum_{n=0}^{\infty}[n,0,n]t^{2n}\,, \tag{3.16}$$

where the SU(4) representation $[n,0,n]$ can be interpreted as the partition function for $\mathcal{N}=6$ chiral multiplets in the Kaluza-Klein (KK) compactification on $\mathbb{P}^3$. It is well known indeed that the KK chiral multiplets for $\mathrm{AdS}_4\times\mathbb{P}^3$ fall in $[n,0,n]$ representations [38].

When restricted to zero SU(4) chemical potentials, we get

$$g_{1,\mathbb{Z}_k}(t,1,1,1;\mathbb{P}^3) = \frac{1+9t^2+9t^4+t^6}{(1-t^2)^6} = \sum_{n=0}^{\infty}\frac{(n+1)^2(n+2)^2(2n+3)}{12}t^{2n}\,, \tag{3.17}$$

where we note that this formula agrees with (3.15) in the limit $k\to\infty$.

It is obvious from the order of the pole that we are dealing with a six dimensional manifold. This is explained by the fact that $\mathbb{Z}_k$ acts by reducing by a factor of k the length of a circle in S^7. In the limit $k\to\infty$, S^7 becomes $\mathbb{P}^3$ and, correspondingly, M-theory is reduced to Type IIA. The above partition function then characterises the protected Type IIA configurations on $AdS_4\times\mathbb{P}^3$ [32]. We point out that this partition function is palindromic even though it is not a CY manifold.

4. Non-abelian orbifold actions on the $N=1$ theory

We now consider the orbifold actions of the binary dihedral,[5] tetrahedral, octahedral and icosahedral discrete subgroups Γ of SU(2) associated to the affine lie algebras $\hat{D}_{k+2}$, $\hat{E}_6$, $\hat{E}_7$, $\hat{E}_8$, whose projections break the Spin(8) R-symmetry group into Sp(2) and preserves $\mathcal{N}=5$ supersymmetry.[6] We note that the membrane theory on $\mathbb{R}^8/\Gamma$ has a dual $AdS_4\times S^7/\Gamma$.

[5] In this paper, we shall denote the binary dihedral group of order $4k$ by $\hat{D}_{k+2}$.

[6] $\mathbb{C}^4/\Gamma$, with Γ discrete subgroup of SU(2) acting diagonally on two copies of $\mathbb{C}^2$, is obviously a Calabi-Yau cone on S^7/Γ. It preserves $\mathcal{N}=6$ supersymmetry for Abelian Γ and $\mathcal{N}=5$ for dihedral and exceptional Γ [18] , as can be checked by the action on spinors. Notice however that in the text we adopted a real notation which is related to the Calabi-Yau complex coordinates by a change of complex structure.

Discrete Molien formula. The partition function for S^7/Γ depending on the parameter t can be easily computed by the following discrete Molien formula (c.f. Equation (3.1) of [8]):

$$g_{1,\Gamma}(t) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{1-t^2}{\det(I_{8\times 8}-t\gamma)}\ , \tag{4.1}$$

where the determinant is taken over the 8×8 matrix representation of the group elements.

Decomposing SU(4) into Sp(2). Since the SU(4) R-symmetry is broken into Sp(2), we will need to expand partition functions in terms of irreducible representations of Sp(2) instead of SU(4). We shall quote here the relevant decomposition formula (setting the fuacities z_i of SU(4) to the fugacities x_i of Sp(2) to be $z_1 = x_1, z_2 = x_2, z_3 = x_1$; this action is like a "folding" of the representation similar to the action of an orientifold plane.):

$$[m,0,n]_{\mathrm{SU}(4)} \to \sum_{a=0}^{\min\{m,n\}} [m+n-2a,a]_{\mathrm{Sp}(2)}\ . \tag{4.2}$$

Therefore, we can rewrite (3.6) in terms of Sp(2) irreducible representations, setting $b=1$ since the baryonic charge is not conserved with non-Abelian orbifold projections,

$$g_{1,\mathbb{Z}_k}(t,x_1,x_2) = \sum_{n_1,n_2=0}^{\infty}\ \sum_{a=0}^{p(n_1,n_2)}\ \sum_{r=0}^{k-1} [k(n_1+n_2)+2(r-a),a]\ t^{k(n_1+n_2)+2r}\ , \tag{4.3}$$

where $p(n_1,n_2) = \min\{kn_1+r, kn_2+r\}$, and $x_1,\ x_2$ are the Sp(2) fugacities.

4.1 $\hat{D}_{k+2}$ orbifolds

Let us consider the group $\hat{D}_{k+2}$ which is a subgroup of SO(8). It is generated by

$$\begin{pmatrix} wI_{4\times 4} & 0 \\ 0 & w^{-1}I_{4\times 4}\end{pmatrix}\quad ,\quad \begin{pmatrix} 0 & iJ_{4\times 4} \\ -iJ_{4\times 4} & 0\end{pmatrix} \tag{4.4}$$

where $w^{2k}=1$ and $J_{4\times 4}$ is the four by four symplectic matrix.[7] The matrices in the previous formula are acting on the vector representation $[1,0,0,0]$ of Spin(8) in a complex notation where it decomposes as a fundamental $[1,0,0]$ plus anti-fundamental $[0,0,1]$ representation of SU(4). The global symmetry group $\mathrm{SU}(4)\times\mathrm{U}(1)_B$ is reduced by the projection to Sp(2), which is simply the group of SU(4) matrices satisfying the condition $Jg = g^*J$.

General partition function for $\hat{D}_{k+2}$. It can be shown [36] that substituting (4.4) into (4.1) gives the partition function for $N=1$ and $\Gamma = \hat{D}_{k+2}$ for an arbitrary k. This substitution and the substitution for the other non-abelian groups is consistent with the formulas in table (3.9) of [8].

$$g_{1,\hat{D}_{k+2}}(t,x_1,x_2) = \frac{1}{2}g_{1,\mathbb{Z}_{2k}}(t,x_1,x_2) + g_{1,\mathbb{Z}_4}(t,x_1,x_2) - \frac{1}{2}g_{1,\mathbb{Z}_2}(t,x_1,x_2)\ . \tag{4.5}$$

[7]These generators are consistent with the generators taken from Equations (3.9) and (3.10) of [8] by taking a 2 by 2 block matrix and composing it with the 4 by 4 matrix that has an identity in the diagonal components, J in the upper block and $-J$ in the lower block. A similar construction follows for the other non-abelian subgroups of SU(2) as stated explicitly below.

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The rationale for this formula is that we can consider $\hat{D}_{k+2}$ as composed of a subgroup $\mathbb{Z}_{2k}$ and k subgroups of $\mathbb{Z}_4$, each with common intersection $\mathbb{Z}_2$. (4.5) is a surgery formula, as in [10], for this decomposition.

An example of $\hat{D}_4$. Substituting $k=2$ into (4.5) and using (3.7), (3.11), we find that the unrefined partition function is given by

$$\begin{aligned} g_{1,\hat{D}_4}(t,1,1) &= \frac{1+2t^2+68t^4+78t^6+214t^8+78t^{10}+68t^{12}+2t^{14}+t^{16}}{(1-t^2)^7(1+t^2)^4} \\ &= 1+5t^2+84t^4+324t^6+1221t^8+3185t^{10}+\cdots\,. \end{aligned} \tag{4.6}$$

The plethystic logarithm is given by

$$g_{1,\hat{D}_4}(t,1,1) = 5t^2+69t^4-56t^6-2019t^8+3368t^{10}+\cdots\,. \tag{4.7}$$

The coefficient 5 of t^2 indicates that there are 5 generators transforming in the $[0,1]$ representation, and the coefficient 69 of t^4 is the dimension of the representation $[4,0]+[2,1]-[0,0]$. We note that the correction [0,0], which is contained in the decomposition of $\mathrm{Sym}^2[0,1]$, simply indicates that there is a relation of order t^4. Thus, the generators of this theory transform in the representations [0,1], [2,1] and [4,0].

General formulae. Substituting (3.15) into (4.5), we obtain the general unrefined partition function for $\hat{D}_{k+2}$:

$$\begin{aligned} g_{1,\hat{D}_{k+2}}(t,1,1) &= \frac{1}{3\left(1-t^2\right)^6\left(1+t^2\right)^4\left(1-t^{2k}\right)^4} \\ &\times\Big[3+9t^2+108t^4+90t^6+195t^8+45t^{10}+30t^{12}+\left(-9+11k+12k^2+4k^3\right)t^{2k} \\ &+\left(9-22k+16k^3\right)t^{4k}+\left(-3+11k-12k^2+4k^3\right)t^{6k}-\big(3+71k+36k^2 \\ &\quad +4k^3\big)t^{6(2+k)}+\left(63+142k-16k^3\right)t^{4(3+k)}+3\left(-9-49k+4k^2+4k^3\right)t^{2(5+k)} \\ &-\left(81+71k-36k^2+4k^3\right)t^{2(6+k)}+\left(3-11k+12k^2-4k^3\right)t^{2(7+k)} \\ &+\left(3+71k+36k^2+4k^3\right)t^{2+2k}-3\left(93-49k-4k^2+4k^3\right)t^{4+2k} \\ &-3\left(25-29k+20k^2+4k^3\right)t^{6+2k}+3\left(-165-29k-20k^2+4k^3\right)t^{8+2k} \\ &+\left(-63-142k+16k^3\right)t^{2+4k}-3\left(-63+98k+16k^3\right)t^{4+4k} \\ &-3\left(105+58k+16k^3\right)t^{6+4k}+3\left(105+58k+16k^3\right)t^{8+4k} \\ &+3\left(-63+98k+16k^3\right)t^{10+4k}+\left(-9+22k-16k^3\right)t^{14+4k} \\ &+\left(81+71k-36k^2+4k^3\right)t^{2+6k}-3\left(-9-49k+4k^2+4k^3\right)t^{4+6k} \\ &+\left(495+87k+60k^2-12k^3\right)t^{6+6k}+3\left(25-29k+20k^2+4k^3\right)t^{8+6k} \\ &+3\left(93-49k-4k^2+4k^3\right)t^{10+6k}-\left(-9+11k+12k^2+4k^3\right)t^{14+6k} \\ &-30t^{2+8k}-45t^{4+8k}-195t^{6+8k}-90t^{8+8k}-108t^{10+8k}-9t^{12+8k}-3t^{14+8k}\Big]\,, \end{aligned} \tag{4.8}$$

where we note that this formula is consistent with the above specific examples. An explicit expression for the refined partition function is given by

$$g_{1,\hat{D}_4}(t,x_1,x_2) = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{j=0,\neq n-1}^{n}[2n+4p-2j,j]t^{2n+4p}. \tag{4.9}$$

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4.2 $\hat{E}_6$ orbifold

Let us consider the group $\hat{E}_6$ which is a subgroup of Spin(8). It is generated by

$$\frac{1}{2}\begin{pmatrix} (-1+i)I_{4\times 4} & (-1+i)J_{4\times 4} \\ -(1+i)J_{4\times 4} & (-1-i)I_{4\times 4} \end{pmatrix} \quad , \quad \begin{pmatrix} iI_{4\times 4} & 0 \\ 0 & -iI_{4\times 4} \end{pmatrix} \tag{4.10}$$

Full partition function for $\hat{E}_6$. It can be shown [36] that the partition function for $N = 1$ and $\Gamma = \hat{E}_6$ is

$$g_{1,\hat{E}_6}(t, x_1, x_2) = g_{1,\mathbb{Z}_6}(t, x_1, x_2) + \frac{1}{2}\left(g_{1,\mathbb{Z}_4}(t, x_1, x_2) - g_{1,\mathbb{Z}_2}(t, x_1, x_2)\right) \ . \tag{4.11}$$

Substituting (4.3) into (4.11), we obtain the full partition function for $\hat{E}_6$.

The unrefined partition function. We obtain a simpler expression if the x's are set to unity:

$$\begin{aligned} g_{1,\hat{E}_6}(t,1,1) &= \frac{1}{(1-t^2)^7\,(1+2t^2+2t^4+t^6)^4} \\ &\quad\times(1+6t^2+16t^4+106t^6+487t^8+996t^{10}+1532t^{12}+2332t^{14}+2872t^{16} \\ &\qquad+2332t^{18}+1532t^{20}+996t^{22}+487t^{24}+106t^{26}+16t^{28}+6t^{30}+t^{32}) \\ &= 1+5t^2+14t^4+114t^6+451t^8+975t^{10}+\cdots \ . \end{aligned} \tag{4.12}$$

The plethystic logarithm is given by

$$g_{1,\hat{E}_6}(t,1,1) = 5t^2 - t^4 + 84t^6 - 24t^8 - 172t^{10} + \cdots \ . \tag{4.13}$$

The coefficient 5 of t^2 indicates that there are 5 generators transforming in the $[0,1]$ representation, the coefficient -1 of t^4 indicates that there is a relation transforming in the trivial representation, and the coefficient 84 of t^4 indicates that there are 84 generators transforming in the $[6,0]$ representation. Thus, the first generators of this theory transform in the representations $[0,1]$ and $[6,0]$.

4.3 $\hat{E}_7$ orbifold

Let us consider the group $\hat{E}_7$ which is a subgroup of Spin(8). It is generated by

$$\frac{1}{2}\begin{pmatrix} (-1+i)I_{4\times 4} & (-1+i)J_{4\times 4} \\ -(1+i)J_{4\times 4} & (-1-i)I_{4\times 4} \end{pmatrix} \quad , \quad \frac{1}{\sqrt{2}}\begin{pmatrix} (1+i)I_{4\times 4} & 0 \\ 0 & (1-i)I_{4\times 4} \end{pmatrix} \tag{4.14}$$

Full partition function for $\hat{E}_7$. It can be shown [36] that the partition function for $N = 1$ and $\Gamma = \hat{E}_7$ is

$$g_{1,\hat{E}_7}(t, x_1, x_2) = \frac{1}{2}(g_{1,\mathbb{Z}_8}(t, x_1, x_2) + g_{1,\mathbb{Z}_6}(t, x_1, x_2) + g_{1,\mathbb{Z}_4}(t, x_1, x_2) - g_{1,\mathbb{Z}_2}(t, x_1, x_2)) \ . \tag{4.15}$$

Substituting (4.3) into (4.15), we obtain the full partition function for $\hat{E}_7$.

The unrefined partition function. We obtain a simpler expression if the x's are set to unity:

$$\begin{aligned}
g_{1,\hat{E}_7}(t,1,1) &= \frac{1}{(1-t^2)^7\,(1+2t^2+3t^4+3t^6+2t^8+t^{10})^4} \\
&\quad \times(1+6t^2+20t^4+46t^6+242t^8+686t^{10}+1921t^{12}+3602t^{14}+6037t^{16} \\
&\qquad +8672t^{18}+11947t^{20}+14252t^{22}+15728t^{24}+14252t^{26}+11947t^{28} \\
&\qquad +8672t^{30}+6037t^{32}+3602t^{34}+1921t^{36}+686t^{38}+242t^{40}+46t^{42} \\
&\qquad +20t^{44}+6t^{46}+t^{48}) \\
&= 1+5t^2+14t^4+30t^6+220t^8+520t^{10}+\cdots\ . \qquad (4.16)
\end{aligned}$$

The plethystic logarithm is given by

$$g_{1,\hat{E}_7}(t,1,1) = 5t^2 - t^4 + 165t^8 - 396t^{10} + \cdots\ . \qquad (4.17)$$

The coefficient 5 of t^2 indicates that there are 5 generators transforming in the $[0,1]$ representation, the coefficient -1 of t^4 indicates that there is a relation transforming in the trivial representation, and the coefficient 165 of t^8 indicates that there are 84 generators transforming in the $[8,0]$ representation. Thus, the first generators of this theory transform in the representations $[0,1]$ and $[8,0]$.

4.4 $\hat{E}_8$ orbifold

Let us consider the group $\hat{E}_8$ which is a subgroup of Spin(8). It is generated by the same generators as $\hat{E}_6$ with the addition of

$$\frac{1}{4}\begin{pmatrix} 2iI_{4\times4} & ((1-\sqrt{5})-i(1+\sqrt{5}))J_{4\times4} \\ ((1-\sqrt{5})+i(1+\sqrt{5}))J_{4\times4} & -2iI_{4\times4} \end{pmatrix} \qquad (4.18)$$

Full partition function for $\hat{E}_8$. It can be shown [36] that the partition function for $N=1$ and $\Gamma=\hat{E}_8$ is

$$g_{1,\hat{E}_8}(t,x_1,x_2) = \frac{1}{2}\left(g_{1,\mathbb{Z}_{10}}(t,x_1,x_2)+g_{1,\mathbb{Z}_6}(t,x_1,x_2)+g_{1,\mathbb{Z}_4}(t,x_1,x_2)-g_{1,\mathbb{Z}_2}(t,x_1,x_2)\right)\ . \qquad (4.19)$$

Substituting (4.3) into (4.19), we obtain the full partition function for $\hat{E}_8$.

The unrefined partition function. We obtain a simpler expression if the x's are set to unity:

$$\begin{aligned}
g_{1,\hat{E}_8}(t,1,1) &= \frac{1}{(1-t^2)^7\,(1+3t^2+5t^4+6t^6+6t^8+5t^{10}+3t^{12}+t^{14})^4} \\
&\quad \times(1+10t^2+50t^4+166t^6+410t^8+798t^{10}+1711t^{12}+4970t^{14} \\
&\qquad +14024t^{16}+30920t^{18}+53137t^{20}+75728t^{22}+97846t^{24}+124794t^{26} \\
&\qquad +160086t^{28}+194598t^{30}+209502t^{32}+194598t^{34}+160086t^{36}+124794t^{38} \\
&\qquad +97846t^{40}+75728t^{42}+53137t^{44}+30920t^{46}+14024t^{48}+4970t^{50}+1711t^{52}
\end{aligned}$$

$$+798t^{54} + 410t^{56} + 166t^{58} + 50t^{60} + 10t^{62} + t^{64})$$
$$= 1 + 5t^2 + 14t^4 + 30t^6 + 55t^8 + 91t^{10} + \cdots \ . \tag{4.20}$$

The plethystic logarithm is given by

$$g_{1,\hat{E}_8}(t,1,1) = 5t^2 - t^4 + 455t^{12} - 1170t^{14} + \cdots \ . \tag{4.21}$$

The coefficient 5 of t^2 indicates that there are 5 generators transforming in the $[0,1]$ representation, the coefficient -1 of t^4 indicates that there is a relation transforming in the trivial representation, and the coefficient 455 of t^{12} indicates that there are 455 generators transforming in the $[12,0]$ representation. Thus, the first generators of this theory transform in the representations $[0,1]$ and $[12,0]$.

5. Higher N theories

Having dealt with various $N = 1$ theories, we turn to the problem of counting operators in higher N case.

The moduli space. We will denote the moduli space for N branes on $\mathbb{R}^8$ by $S^N(\mathbb{R}^8)$. Restricting to S^7, we quotient this out by the non-compact direction $\mathbb{R}^+$ to get $\mathcal{M}_N(S^7) = S^N(\mathbb{R}^8)/\mathbb{R}^+$. This moduli space has a real dimension $8N - 1$.

The grand canonical partition function. We use the plethystic exponential and write down the generating function for higher values of N by introducing a fugacity ν for the number of M2-branes. We may also choose to count operators on S^7. For this purpose we first write the grand canonical partition function for $\mathbb{R}^8$,

$$g(\nu; t, y_1, y_2, y_3, y_4; \mathbb{R}^8) = \mathrm{PE}\left[\nu g_{1,\Gamma}(t, y_1, y_2, y_3, y_4; \mathbb{R}^8)\right], \tag{5.1}$$

that has an expansion in terms of partition functions for a fixed number of branes

$$g(\nu; t, y_1, y_2, y_3, y_4; \mathbb{R}^8) = \sum_{N=0}^{\infty} \nu^N g_{N,\Gamma}\left(t, y_1, y_2, y_3, y_4; S^N(\mathbb{R}^8)\right) \tag{5.2}$$

Explicitly, the formulae for the first few N are as follows:

$$\begin{aligned} g_{2,\Gamma}(t, y; S^2(\mathbb{R}^8)) &= \frac{1}{2}\left[g_{1,\Gamma}(t, y; \mathbb{R}^8)^2 + g_{1,\Gamma}(t^2, y^2; \mathbb{R}^8)\right], \\ g_{3,\Gamma}(t, y; S^3(\mathbb{R}^8)) &= \frac{1}{6}\left[g_{1,\Gamma}(t, y; \mathbb{R}^8)^3 + 3g_{1,\Gamma}(t, y) g_{1,\mathbb{Z}_k}(t^2, y^2; \mathbb{R}^8) + 2g_{1,\Gamma}(t^3, y^3; \mathbb{R}^8)\right], \end{aligned} \tag{5.3}$$

where we have written $y_1,\ y_2,\ y_3,\ y_4$ collectively as y.

Operators on S^7. The projection to protected operators is more complicated than for the $N = 1$ case since products of short multiplets are not necessarily short. We just remove an overall trace and regard these partition functions as counting real functions on the moduli space for N branes. One needs to note that *the restriction to a fixed radius should be done only once and should not be symmetrised over.* We get the reduced grand canonical partition function,

$$g_{\Gamma}(\nu; t, y_1, y_2, y_3, y_4; S^7) = (1 - t^2)\mathrm{PE}\left[\nu g_{1,\Gamma}(t, y_1, y_2, y_3, y_4; \mathbb{R}^8)\right]. \tag{5.4}$$

An example of $N = 2$ and $k = 1$. Using the first formula in (5.3) together with (5.4), we find that the coefficient of ν^2 in the above expression is

$$\begin{aligned} g_{2,\mathbb{Z}_1}(t,y;\mathcal{M}_2(S^7)) = & \frac{1-t^2}{2\left(1-\frac{t}{y_1}\right)^2(1-ty_1)^2\left(1-\frac{ty_1}{y_2}\right)^2\left(1-\frac{ty_2}{y_1}\right)^2\left(1-\frac{ty_2}{y_3y_4}\right)^2\left(1-\frac{ty_3}{y_4}\right)^2} \\ & \times\frac{1}{\left(1-\frac{ty_4}{y_3}\right)^2\left(1-\frac{ty_3y_4}{y_2}\right)^2} \\ & +\frac{1-t^2}{2\left(1-\frac{t^2}{y_1^2}\right)(1-t^2y_1^2)\left(1-\frac{t^2y_1^2}{y_2^2}\right)\left(1-\frac{t^2y_2^2}{y_1^2}\right)\left(1-\frac{t^2y_2^2}{y_3^2y_4^2}\right)\left(1-\frac{t^2y_3^2}{y_4^2}\right)} \\ & \times\frac{1}{\left(1-\frac{t^2y_4^2}{y_3^2}\right)\left(1-\frac{t^2y_3^2y_4^2}{y_2^2}\right)}\,. \end{aligned} \tag{5.5}$$

This can be expanded in terms of irreducible representations of Spin(8) as

$$g_{2,\mathbb{Z}_1}(t,y;\mathcal{M}_2(S^7)) = 1+(2[2,0,0,0]+1)\,t^2+(2[3,0,0,0]+2[1,0,0,0]+[1,1,0,0])\,t^3+\cdots\,. \tag{5.6}$$

Note that at order 2 we again find the singlet operator of the form $\mathrm{Tr}(\phi_i\phi_i)$ or $\mathrm{Tr}(\phi_i)\mathrm{Tr}(\phi_i)$, either of which is unprotected. This is the simplest example which demonstrates that higher N generating functions do not count protected operators. When setting the Spin(8) chemical potentials to zero we find

$$\begin{aligned} g_{2,\mathbb{Z}_1}(t,1,1,1,1;\mathcal{M}_2(S^7))) &= \frac{1+28t^2+70t^4+28t^6+t^8}{(1-t)^{15}(1+t)^7} \\ &= 1+8t+71t^2+400t^3+1884t^4+7344t^5+\cdots \end{aligned} \tag{5.7}$$

with the pole of order 15 at $t = 1$ indicating that the reduced moduli space is indeed $8 \times 2 - 1 = 15$ real dimensional. The plethystic logarithm of this expression is

$$\mathrm{PL}[g_{2,\mathbb{Z}_1}(t,1,1,1,1;\mathcal{M}_2(S^7)))] = 8t+35t^2-336t^4+5376t^6-101856t^8+\cdots\,. \tag{5.8}$$

This indicates that there are 8 generators transforming in the Spin(8) representation $[1,0,0,0]$ at order t, 35 generators transforming in $[2,0,0,0]$ at order t^2, and 336 relations transforming in $[0,2,0,0]+[2,0,0,0]+[0,0,0,0]$ at order t^6.

An example of $N = 2$ and $k = 2$. For simplicity, let us work with unrefined partitions. Starting from the case of $N = 1$, we have

$$g_{1,\mathbb{Z}_2}(t,1,1,1,1;\mathbb{R}^8) = \frac{g_{1,\mathbb{Z}_2}(t,1,1,1,1;S^7)}{1-t^2} = \frac{1+28t^2+70t^4+28t^6+t^8}{(1-t^2)^8}\,, \tag{5.9}$$

where we have used (3.7) in the second equality. Using the first formula in (5.3) and restricting to S^7, we find that

$$\begin{aligned} g_{2,\mathbb{Z}_2}(t,1,1,1,1;\mathcal{M}_2(S^7)) =& \frac{1}{(1-t^2)^{15}\,(1+t^2)^8}\times(1+28t^2+728t^4+6356t^6+34140t^8 \\ & +110300t^{10}+254184t^{12}+403508t^{14}+478662t^{16}+403508t^{18} \\ & +254184t^{20}+110300t^{22}+34140t^{24}+6356t^{26}+728t^{28}+28t^{30}+t^{32}) \\ =&\ 1+35t^2+960t^4+12600t^6+109230t^8+\cdots\,. \end{aligned} \tag{5.10}$$

The plethystic logarithm of this expression is

$$\mathrm{PL}[g_{2,\mathbb{Z}_2}(t,1,1,1,1;\mathcal{M}_2(S^7)))] = 35t^2 + 330t^4 - 6720t^6 + 8100t^8 + \cdots \,. \tag{5.11}$$

This indicates that there are 35 generators transforming in the Spin(8) representation $[2,0,0,0]$ at order t^2, and 330 generators transforming in $[4,0,0,0]+[2,0,0,0]+[0,0,0,0]$ at order t^4.

The palindromic property. Note that the previous partition functions are palindromic. This happened for all the partition functions for one or more membranes that we encountered in this paper. It can be explained as follows. Recall that the palindromic property characterizes Calabi-Yau (Gorenstein) singularities [15]. Although we are considering real coordinates, the partition function for a membrane on $\mathbb{R}^8/\Gamma$ can be equivalently considered as a partition function for holomorphic functions on the complexification $\mathbb{C}^8/\Gamma$ which is indeed a non-compact Calabi-Yau singularity. Analogously, for N membranes, we deal with the symmetric product of N Calabi-Yau four-folds which is also a non-compact Calabi-Yau singularity.[8]

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A. Character computations: plethystic amusement

In this section we present an efficient method for computing all characters of a given group. This method is good when the rank of the group is small enough. We will demonstrate this method for the group SO(5), with Sp(2) characters being obtained as by-products. Moreover, SU(4) characters will be mentioned at the end of this section. This method can be repeated for the group SO(8).

SO(5) characters. Let us start by looking at a generic representation of SO(5) of the form $[n_1,n_2]_{\mathrm{SO}(5)}$ with dimension formula [39] given by

$$\dim[n_1,n_2]_{\mathrm{SO}(5)} = \frac{(n_1+1)(n_2+1)(n_1+n_2+2)(2n_1+n_2+3)}{6}. \tag{A.1}$$

[8]This should be contrasted with the three dimensional case where moduli spaces for $N>1$ are not Calabi-Yau [15] since symmetrized products of odd dimensional Calabi-Yaus are not.

We can introduce a formal generating function

$$g_{\mathrm{SO}(5)}(t_1,t_2;w_1,w_2)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}[n_1,n_2]_{\mathrm{SO}(5)}t_1^{n_1}t_2^{n_2}\ , \tag{A.2}$$

where t_1, t_2 are weights which keep track with the highest weight representation of the group, while w_1, w_2 are the SO(5) fugacities. The function $g_{\mathrm{SO}(5)}$ can be summed easily. One can write down the expression when the SO(5) chemical potentials are set to zero,

$$g_{\mathrm{SO}(5)}(t_1,t_2;1,1)=\frac{1-t_1^2-4t_1t_2+t_1t_2^2+4t_1^2t_2-t_2^2t_1^3}{(1-t_1)^5(1-t_2)^4}, \tag{A.3}$$

and immediately realize that this can be written in terms of the simple representations of SO(5),

$$g_{\mathrm{SO}(5)}(t_1,t_2;w_1,w_2)=(1-t_1^2-[0,1]t_1t_2+t_1t_2^2+[0,1]t_1^2t_2-t_1^3t_2^2)\mathrm{PE}\left[[1,0]t_1+[0,1]t_2\right]\ , \tag{A.4}$$

One can compute the character of the representation with highest weights $[n_1, n_2]$ as the coefficient of the $t_1^{n_1}t_2^{n_2}$ term in the power expansion of $g_{\mathrm{SO}(5)}$. For completeness we record here a possible explicit form for the two representations which are needed to evaluate this expansion,

$$[1,0]_{\mathrm{SO}(5)}=\frac{w_2^2}{w_1}+w_1+\frac{1}{w_1}+1+\frac{w_1}{w_2^2},\quad [0,1]_{\mathrm{SO}(5)}=\frac{w_1}{w_2}+w_2+\frac{1}{w_2}+\frac{w_2}{w_1}\ . \tag{A.5}$$

As an example, let us consider the SO(5) representation $[1, 1]$. Equation (A.2) suggests that the character of $[1, 1]$ is simply the coefficient of t_1t_2 in the power expansion of the right hand side of (A.4). Substituting (A.5) into (A.4) and expanding it as a power series, we find that the coefficient of t_1t_2 is

$$[1,1]_{\mathrm{SO}(5)}=\frac{w_1}{w_2^3}+\frac{w_1^2}{w_2^3}+\frac{2}{w_2}+\frac{1}{w_1w_2}+\frac{2w_1}{w_2}+\frac{w_1^2}{w_2}+2w_2+\frac{w_2}{w_1^2}+\frac{2w_2}{w_1}+w_1w_2+\frac{w_2^3}{w_1^2}+\frac{w_2^3}{w_1}\ . \tag{A.6}$$

Sp(2) characters: by-products. The dimension formula [39] of the representation $[n_1, n_2]_{\mathrm{Sp}(2)}$ is given by

$$\dim[n_1,n_2]_{\mathrm{Sp}(2)}=\frac{(n_1+1)(n_2+1)(n_1+n_2+2)(n_1+2n_2+3)}{6}\ . \tag{A.7}$$

Observe that this is simply the interchange of n_1 and n_2 in formula (A.1), i.e.

$$\dim[n_1,n_2]_{\mathrm{Sp}(2)}=\dim[n_2,n_1]_{\mathrm{SO}(5)}\ . \tag{A.8}$$

Therefore, the corresponding (A.3) for Sp(2) is

$$g_{\mathrm{Sp}(2)}(t_1,t_2;1,1)=g_{\mathrm{SO}(5)}(t_2,t_1;1,1)\ . \tag{A.9}$$

We note that $\dim[1,0]_{\mathrm{Sp}(2)}=4$ and $\dim[0,1]_{\mathrm{Sp}(2)}=5$. According to (A.4), we have

$$\begin{aligned} g_{\mathrm{Sp}(2)}(t_1,t_2;w_1,w_2) &= (1-t_2^2-[1,0]t_2t_1+t_2t_1^2+[1,0]t_2^2t_1-t_2^3t_1^2)\mathrm{PE}\left[[1,0]t_2+[0,1]t_1\right] \\ &= g_{\mathrm{SO}(5)}(t_2,t_1;w_2,w_1)\ . \end{aligned} \tag{A.10}$$

Thus, we arrive at an amusing relation between irreducible representations of Sp(2) and SO(5):

$$[n_1,n_2]_{\mathrm{Sp}(2)}(w_1,w_2)=[n_2,n_1]_{\mathrm{SO}(5)}(w_2,w_1)\ . \tag{A.11}$$

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SU(4) characters. By a similar process as above, we find the following generating function for SU(4):

$$\begin{aligned} g_{\mathrm{SU}(4)}(t_1,t_2,t_3;z_1,z_2,z_3) = &\ (1-[0,0,1]t_1t_2 - t_1t_3 - t_2^2 - [1,0,0]t_2t_3 \\ &+t_1^2t_2 + [1,0,0]t_1t_2^2 + [0,1,0]t_1t_2t_3 + [0,0,1]t_2^2t_3 + t_2t_3^2 \\ &-t_1^2t_2^3 - [0,0,1]t_1t_2^2t_3^2 - [0,1,0]t_1t_2^3t_3 - [1,0,0]t_1^2t_2^2t_3 - t_2^3t_3^2 \\ &+[0,0,1]t_1^2t_2^3t_3 + t_1^2t_3^2t_2^2 + t_1t_2^4t_3 + [1,0,0]t_1t_2^3t_3^2 - t_1^2t_2^4t_3^2) \\ &\times \mathrm{PE}\left[[1,0,0]t_1 + [0,1,0]t_2 + [0,0,1]t_3\right] \ . \end{aligned} \tag{A.12}$$

The character of the irreducible representation $[n_1,n_2,n_3]_{\mathrm{SU}(4)}$ can be simply read out from the coefficient of $t_1^{n_1}t_2^{n_2}t_3^{n_3}$ in the power series of this expression.

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